$R-$positivity of matrices and Hamiltonians on nearest neighbors trajectories

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Abstract. *We revisit the $R-$positivity of nearest neighbors matrices on $\mathbb{Z}_+$ and the Gibbs measures on the set of nearest neighbors trajectories on $\mathbb{Z}_+$ whose Hamiltonians award either visits to sites a or visits to edges. We give conditions that guarantee the $R-$positivity or equivalently the existence of the infinite volume Gibbs measure, and we show geometrical recurrence of the associated Markov chain. In this work we generalize and sharpen results obtained in [3] and [6].*

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1 $R-$ positivity and main result

Let $Q = (q_{x,y} : x, y \in \mathbb{Z}_+)$ be a nearest neighbors matrix on $\mathbb{Z}_+$, i.e.

$$q_{x,y} = 0 \text{ if } |x-y| \neq 1 \text{ and } q_{x,y} > 0 \text{ if } |x-y| = 1. \tag{1}$$

From irreducibility we get that

$$R(Q) = \left(\limsup_{N\to\infty} \left(q_{x,x}^{(2N)} \right)^{1/2N} \right)^{-1},$$

is a common convergence radius, i.e. it is independent of $x \in \mathbb{Z}_+$. In this work we will assume $R(Q) > 0$. Let us put $R = R(Q)$.

The matrix Q is said to be R-recurrent if $\sum\limits_{n\geq 0} q_{x,x}^{(2N)} R^n = \infty$, otherwise it is called R-transient. Consider the continued fraction

$$H(Q,R) := \cfrac{q_{0,1}q_{1,0}/R^2}{1 - \cfrac{q_{1,2}q_{2,1}/R^2}{\cfrac{\cdots}{1 - \cfrac{q_{x,x+1}q_{x+1,x}/R^2}{\cdots}}}}.$$

For the class of matrices Q of type (1), it was shown in ([2]) that Q is $R-$recurrent if and only if $H(Q,R)=1$ and it is $R-$transient when $H(Q,R)<1$ (the proof uses strongly Theorem 11.2 of Wall in [10]). We have that there exists a solution to the eigenvector problem:

$$Q\vec{f}=R^{-1}\vec{f} \text{ with } \vec{f}=(f_x : x\in\mathbb{Z}_+)>0\,.$$

(For general positive matrices the existence of a solution is only guaranteed for $R-$recurrent matrices). Note that the matrix $P^{(R)}=(p_{x,y}:x,y\in\mathbb{Z}_+)$ defined by

$$p_{x,y}=Rq_{x,y}\frac{f_y}{f_x}\quad x,y\in\mathbb{Z}_+,$$

is an stochastic matrix of a birth and death chain $X^Q=(X^Q_n:n\geq 0)$ reflected at 0. The matrix Q is said to be R-positive recurrent if $P^{(R)}$ is a positive recurrent Markov chain. (For the previous definitions and results see Vere-Jones in [8] and [9]).

For $x\geq 0$ we denote by $\tau_x := \inf\{n>0 : X^Q_n = x\}$ the return time to x, where as usual we put $\tau_x=\infty$ when $X^Q_n\neq x$ for all $n>0$.

Denote $J^{[m]}=\{x\in\mathbb{Z}_+ : x\geq m\}$. Let us consider the family of matrices

$$Q^{[m]}=(q_{x,y}:x,y\in J^{[m]})\,,\ m\geq 0\,. \tag{2}$$

With this notation $Q^{[0]}=Q$ the original matrix. Denote by $R^{[m]}:=R(Q^{[m]})$ the convergence radius of $Q^{[m]}$, and by $X^{[m]}:=X^{Q^{[m]}}=(X^{[m]}_n:n\geq 0)$ the associated stochastic matrix. By definition the sequence $(R^{[m]}:m\geq 0)$ is increasing:

$$\forall m\geq 0\ :\qquad R^{[m]}\leq R^{[m+1]}\,,$$

so we can only have two situations:

• The sequence is constant: $R^{[0]}=R^{[m]}$, $\forall m\geq 0$;

• The sequence has a gap, so there exists some $m\geq 0$ such that $R^{[m]}<R^{[m+1]}$.

Our main result is:

Theorem 1 *Let Q matrix of type (1). Assume $R(Q)>0$. If there exists a gap, $R^{[m]}<R^{[m+1]}$ for some $m\geq 0$, then the matrix Q is $R-$positive. Moreover the associated birth and death chain X^Q is geometrically recurrent:*

$$\forall\, 1<\theta\leq {R^{[m+1]}}^2/{R^{[m]}}^2\ :\qquad \mathbb{E}_x(\theta^{\tau_x})<\infty\,.$$

When there does not exists a gap, $R^{[0]}=R^{[m]}$ for all $m\geq 0$, then the matrix $Q^{[1]}$ is $R-$transient.

We point our that this result generalizes the result on [6] in two directions: the matrix Q is not necessarily substochastic as is the case in [6] and on the other hand we can sharpen $R-$positivity to geometrically recurrence of the associated Markov chain. The tools of this work are close to those used in [3].

2 Hamiltonians and Gibbs measures

Let us put our result in the context of one-dimensional Gibbs measures. We consider the space of trajectories Ω of a nearest neighbors non negative random walk,

$$\Omega = \{w \in \mathbb{Z}_+^{\mathbb{Z}} : |w_i - w_{i+1}| = 1 \ \forall i \in \mathbb{Z}\}.$$

For $i \leq j$ we put $w[i,j] = (w_i, \ldots, w_j)$ and $\Omega[i,j] = \{\sigma = w[i,j] : w \in \mathbb{Z}_+^{\mathbb{Z}}\}$. Let $\vec{\alpha} = (\alpha_x : x \in \mathbb{Z}_+)$, $\vec{b} = (b_x : x \in \mathbb{Z}_+)$, $\vec{c} = (c_x : x \in \mathbb{Z}_+)$ be fixed sequences (the rewards). For $x, y \in \mathbb{Z}_+$ and a block $\sigma \in \Omega[i,j]$ we denote by

$$\mathcal{N}_x(\sigma) = \sum_{k=i}^{j} \delta(\sigma_k, x) \text{ and } \mathcal{N}_{x,y}(\sigma) = \sum_{k=i}^{j-1} \delta(\sigma_k \sigma_{k+1}, xy),$$

the number of times σ visits x and the number of times σ passes through the edge xy respectively, this last quantity vanishing if $|x-y| \neq 1$. The Hamiltonians that respectively award the number of visits to the sites or to the edges, are the following ones on the interval $[i,j]$. For $w \in \Omega$,

$$\begin{aligned} H^{\vec{\alpha}}_{[i,j]}(w) &= \sum_{x \in \mathbb{Z}_+} \alpha_x \mathcal{N}_x(w[i,j+1]) \text{ and} \\ H^{\vec{b},\vec{c}}_{[i,j]}(w) &= \sum_{x \in \mathbb{Z}_+} \Big(b_x \mathcal{N}_{x,x+1}(w[i-1,j+1]) + c_x \mathcal{N}_{x+1,x}(w[i-1,j+1]) \Big). \end{aligned}$$

For the Hamiltonians $H = H^{\vec{\alpha}}$ and $H = H^{\vec{b},\vec{c}}$, the probability measures associated to them are (see [4] Definition 2.9)

$$\mu^H_{[i,j]}(w)(\sigma) = (Z_{[i,j]}(w))^{-1} \sum_{\substack{w'[i-1,j+1] : w'[k,\ell] = \sigma \\ w'_{i-1} = w_{i-1}, w'_{j+1} = w_{j+1}}} e^{H_{[i,j]}(w')}, \quad \sigma \in \Omega[k,\ell],$$

where $i < k, \ell < j$ and $Z_{[i,j]}(w) = \sum_{\substack{w'[i-1,j+1]: \\ w'_{i-1} = w_{i-1}, w'_{j+1} = w_{j+1}}} e^{H_{[i,j]}(w')}$ is the partition function of $w \in \Omega$ in $[i,j]$. We ask for the conditions on $\vec{\alpha}$, or in $\vec{b}$ and $\vec{c}$, such that the Hamiltonians $H^{\vec{\alpha}}$, or $H^{\vec{b},\vec{c}}$, define translational invariant Gibbs measures.

These Hamiltonians are related by the following equalities shown in [3]: when the sequences $\vec{\alpha}$, $\vec{b}$, $\vec{c}$ verify $\alpha_x + \alpha_{x+1} = b_x + c_x$ for $x \in \mathbb{Z}_+$ then there exists a real function $\gamma(n, m, p)$ defined in $\mathbb{Z}_+^3$ such that

$$H^{\vec{b},\vec{c}}_{[i,j]}(w) = H^{\vec{\alpha}}_{[i,j]}(w) + \gamma(w_{i-1}, w_{j+1}, j - i).$$

In particular this relation implies

$$\mu^{H^{\vec{b},\vec{c}}}_{[i,j]}(w)(\sigma) = \mu^{H^{\vec{\alpha}}}_{[i,j]}(w)(\sigma) \text{ for any } \sigma \in \Omega[k, \ell], \text{ with } [k, \ell] \subseteq [i + 1, j - 1].$$

From this result we can restrict ourselves to analyze when the Hamiltonian $H^{\vec{b},\vec{c}}$ defines an infinite volume Gibbs measure because we can always fit $\vec{b}$ and $\vec{c}$ to have $\alpha_x + \alpha_{x+1} = b_x + c_x$ for $x \in \mathbb{Z}_+$. Associated to the Hamiltonian $H^{\vec{b},\vec{c}}$ is the transfer matrix $Q = (q_{x,y} : x, y \in \mathbb{Z}_+)$ of type (1) where $q_{x,x+1} = e^{b_x}$ and $q_{x+1,x} = e^{c_x}$ for $x \geq 0$; and $q_{x,y} = 0$ otherwise. We have,

$$\mu^{H^{\vec{b},\vec{c}}}_N(w)(\sigma) = \frac{Q^{N-k+1}(w_{-(N+1)}, \sigma_{-k}) \prod_{i=-k}^{k-1} Q(\sigma_i, \sigma_{i+1}) Q^{N-k+1}(\sigma_k, w_{N+1})}{Q^{2N+2}(w_{-(N+1)}, w_{N+1})} .$$

From Theorem 1 in Kesten [7] for strictly positive matrices and extended in Theorem C in [5] for irreducible matrices, we have that there exists a unique translational invariant Gibbs state for the Hamiltonian $H^{\vec{b},\vec{c}}$ if and only if Q is a R-positive matrix.

Therefore Theorem 1 give a sufficient condition for the general case of $\vec{b}$ and $\vec{c}$ in order that there exists a unique translational invariant Gibbs state for the Hamiltonian $H^{\vec{b},\vec{c}}$. This result goes beyond the cases analyzed in [3]. We recall that when $\vec{b} + \vec{c}$ is constant for x sufficiently large (that is the sequence is ultimately constant) in Theorems 1.2 and Theorem 1.4 in [3], it was given necessary and sufficient explicit conditions for the existence of a Gibbs measure in terms of $\vec{b} + \vec{c}$. For awards on sites and when $\alpha_{x+2} \leq \alpha_x$ for x sufficiently large, in [3] there was supplied sufficient conditions for the existence of a Gibbs measure. All these conditions were written in terms of continued fractions. We finally point out that in [3] it was also discussed the relations between the results on Hamiltonians awarding the visits to sites with the entropic repulsion of a wall and with the SOS model.

3 Proof of the main result

Let $\vec{f} = (f_x : x \in \mathbb{Z}_+)$. We consider the general eigenvalue problem for matrices of type (1):

$$Q\vec{f} = r^{-1}\vec{f} \text{ for } r > 0,\ \vec{f} > 0\,. \tag{3}$$

For $r \in [R, \infty)$ there is a unique, up to a homothetic transformation, $\vec{f} > 0$ verifying (3). Moreover if for some $r > 0$ there exists a solution to (3) then necessarily $r \in [R, \infty)$ (see [3]).

Let $r \in [R, \infty)$. The matrix $P^{(r)} = (p_{x,y} : x, y \in \mathbb{Z}_+)$ defined by

$$p_{x,y} = r \frac{f_y}{f_x} q_{x,y} \text{ for } x, y \in \mathbb{Z}_+^* \tag{4}$$

is a stochastic matrix of a birth-death chain reflected at 0. We have that $P^{(r)}$ is transient for all $r \in (R, \infty)$.

Let us put

$$\omega_x \doteq p_{x,x+1} = r \frac{f_{x+1}}{f_x} q_{x,x+1}.$$

From (4) the sequence $(w_x : x \in \mathbb{Z}_+)$ verifies the equation:

$$\omega_0 = 1 \text{ and } \omega_{x+1} = 1 - \frac{r^2 q_{x,x+1} q_{x+1,x}}{\omega_x} \text{ for } x \in \mathbb{Z}_+. \tag{5}$$

Conversely it is direct to prove that if the sequence $\vec{\omega} = (\omega_x : x \in \mathbb{Z}_+)$ given by the evolution (5) verifies $\vec{\omega} > 0$, then $\vec{f}$ defined by $f_0 > 0$ and $f_{x+1} = f_0 r^{x+1} \prod_{y=0}^{x} \frac{\omega_y}{q_{y,y+1}}$ for $x \in \mathbb{Z}_+$, verifies (3).

At this point it is convenient to introduce some new notation and a definition. First, for $a > 0$ we consider the following continuous and onto strictly increasing function $\varphi_a : (0, \infty] \to (-\infty, 1]$,

$$\varphi_a(\omega) = 1 - \frac{a}{\omega}.$$

Definition 2.1. Let $\vec{a} = (a_x > 0 : x \in \mathbb{Z}_+) > 0$ be a strictly positive fixed sequence. It is said to be allowed if it verifies

$$\forall x \in \mathbb{Z}_+ : \quad \varphi_{a_x} \circ \cdots \circ \varphi_{a_0}(1) > 0\,.$$

Observe that the inverse $\varphi_a^{-1}(\omega) = \frac{a}{1-\omega}$ satisfies analogous properties as φ_a. Also from the definition we get

$$\text{if } \omega > 0 \text{ and } \varphi_a(\omega) > 0 \text{ then } \varphi_a(\omega) \in (0,1).$$

The first part (i) of the next result was already proven in [1] and the parts (ii) and (iii) were shown in [3].

Lemma 2 *Let $\vec{a} > 0$ be a strictly positive sequence.*

(i) $\vec{a}$ is allowed if and only if it verifies

$$\forall x \in \mathbb{Z}_+,\ \forall y \geq x : \varphi_{a_x}^{-1} \circ \cdots \circ \varphi_{a_y}^{-1}(0) < 1. \tag{2.7}$$

(ii) Let $\vec{d} > 0$ be a strictly positive sequence, then

$$\vec{d} \le \vec{a} \text{ and } \vec{a} \text{ is allowed implies } \vec{d} \text{ is allowed}\,.$$

(iii) Let us denote

$$s\vec{a} = (sa_x : x \in \mathbb{Z}_+) \text{ for } s > 0 \text{ and } \mathcal{I}(\vec{a}) = \{s > 0 : s\vec{a} \text{ is allowed }\}.$$

Then $\mathcal{I}(\vec{a}) = \emptyset$ or $\mathcal{I}(\vec{a}) = (0, s^*]$ for some $s^* \in (0, \infty)$.

As a Corollary to this Lemma and by using [1] and [3], we find that when the sequence $\vec{a}$ verifies $a_x = q_{x,x+1}q_{x+1,x}$ then $s^* = R(Q)^2$ (here $q_{x,y}$ are the coefficients of the matrix Q of type (1)). In this case $\mathcal{I}(\vec{a}) \neq \emptyset$ if and only if $R(Q) > 0$.

Assume $\vec{a}$ is allowed then $\varphi^{-1}_{a_{y+1}}(0) \in (0,1)$ and by the increasing property we get

$$h_{\vec{a}}(x,y) = \varphi^{-1}_{a_x} \circ \cdots \circ \varphi^{-1}_{a_y}(0) < \varphi^{-1}_{a_x} \circ \cdots \circ \varphi^{-1}_{a_{y+1}}(0) = h_{\vec{a}}(x, y+1).$$

i.e. the sequence $h_{\vec{a}}(x,y)$ is strictly increasing in $y \in \mathbb{Z}_+^*$. Then the following limit exists and verifies:

$$h_{\vec{a}}(x,\infty) = \lim_{y \nearrow \infty} h_{\vec{a}}(x,y) \le 1\,.$$

Observe that

$$h_{\vec{a}}(x,y) = \cfrac{a_x}{1 - \cfrac{a_{x+1}}{1 - \cfrac{a_{x+2}}{\cfrac{\cdots}{1 - a_y}}}}\,,$$

then $h_{\vec{a}}(x,\infty)$ is a continued fraction.

Recall the notation (2.9), $\omega_0 = 1$, $\omega_{x+1} = \varphi_{a_x} \circ \cdots \circ \varphi_{a_0}(1)$ for $x \in \mathbb{Z}_+$. Assume $\vec{a}$ is a fixed sequence. We put,

$$\forall s \in (0, s^*] : \qquad h(s; x, y) := h_{s\vec{a}}(x,y) \text{ and } h(s; x, \infty) := h_{s\vec{a}}(x, \infty)\,.$$

Recall the notation $J^{[m]} = \{x \in \mathbb{Z}_+ : x \ge m\}$. Let us consider the family of shifted sequences

$$\vec{a}^{[m]} = (a_x : x \in J^{[m]})\,, \quad m \ge 0\,,$$

and the associated values,

$$s^{*[m]} = \sup\{s \in \mathcal{I}(\vec{a}^{[m]})\}\,, \quad m \ge 0\,.$$

From definition this sequence is increasing: $s^{*[m]} \le s^{*[m+1]}$ for all $m \ge 0$ and so we can only have that there are two possibilities: the sequence is constant i.e. $s^{*[0]} = s^{*[m]}$ $\forall m \ge 0$; or the sequence has a gap, that is there exists some $m \ge 0$ such that $s^{*[m]} < s^{*[m+1]}$.

Lemma 3 *If* $s^{*[m]} < s^{*[m+1]}$ *then* $h(s^{*[m]}; m, \infty) = 1 > h(s^{*[m+1]}; m, \infty)$.

Proof: From $s^{*[m]} < s^{*[m+1]}$ we get $h(s^{*[m]}; m, \infty) \leq 1$ and $h(s^{*[m+1]}; m, \infty) > 1$. The last relation follows because $h(s^{*[m+1]}; m, \infty) \leq 1$ would imply $s^{*[m]} = s^{*[m+1]}$, a contradiction. So we only left to prove that under the assumption we have $h(s^{*[m]}; m, \infty) = 1$.

We claim that:

• $h(s; m, \infty) < 1$ then $h(s; m, \infty)$ is an increasing and continuous function in $s \in \Delta_m := [s^{*[m]}, s^{*[m+1]}]$.

Let us show the increasing part of the claim. Since $0 < h(s; m, y) < 1$ then it is increasing in $y \geq m$. On the other hand for every fixed y we have that for all $s < s'$ with $s, s' \in \Delta_m$, we have $0 < h(s; m, y) < h(s'; m, y) < 1$. Then by taking $\lim_{y \to \infty}$ in this inequality we conclude $h(s; m, \infty) \leq h(s'; m, \infty)$.

Now, let show the continuity part of the claim. Under the assumption, for $s < s'$ with $s, s' \in \Delta_m$, we get that the inequality

$$\begin{aligned} |h(s'; m, \infty) - h(s; m, \infty)| \;\leq\; & |h(s'; m, \infty) - h(s'; m, y)| + |h(r; m, y) - h(s'; m, y)| \\ & + |h(s; m, \infty) - h(s; m, y)|\,, \end{aligned}$$

is verified for all fixed $y \geq m$.

Let $\epsilon > 0$ be fixed. We can find $y \geq m$ sufficiently big such that $|h(s'; m, \infty) - h(s'; m, y)| < \epsilon/3$ and $|h(s; m, \infty) - h(s; m, y)| < \epsilon/3$. Let us fix one of such y. Since $h(s; x, y)$ is continuous when $0 < h(r; x, y) < 1$, there exists $\delta > 0$ such that $0 < |s - s'| < \delta$, $s, s' \in \Delta_m$, implies $|h(s; x, y) - h(s'; x, y)| < \epsilon/3$. The continuity is verified.

From the claim it results that $h(s^{*[m]}; m, \infty) = 1$. Indeed, in the contrary we should have $h(s^{*[m]}; m, \infty) < 1$ and $h(s^{*[m+1]}; 0, \infty) > 1$. Since $h(s; m, \infty)$ is increasing and continuous there would exist $s' \in (s^{*[m]}, s^{*[m+1]})$ such that $h(s'; m, \infty) = 1$. Since $h(s'; m, \infty) = \lim\limits_{y \to \infty} a_0 s'/(1 - h(s'; 1, y)$, we should get $0 < h(s'; m, y) < 1\ \forall y \geq m$ and so $s' > s^{*[m]}$ is such that $s'\vec{a}$ is allowed, which contradicts the maximality property satisfied by $s^{*[m]}$. $\Box$

Let us study the R-positivity in Theorem 1, so we are under the hypothesis $R(Q) > 0$. We fix the sequence

$$a_x = q_{x,x+1} q_{x+1,x}\,,\ x \geq 0\,.$$

For the matrices $Q^{[m]}$ defined in (2) we consider the sequences $\vec{a}^{[m]}$ already defined and the reflected birth and death chain $X^{[m]} = (X_n^{[m]} : n \geq 0)$ taking values in $J^{[m]}$ with transition matrices,

$$\omega_m^{[m]} = 1 \text{ and } \omega_{x+1}^{[m]} = 1 - \frac{{R^{[m]}}^2 a_x}{\omega_x^{[m]}},\ x \in J^{[m]}\,, \tag{6}$$

(see (5)). The transition probabilities of the birth and death chain $X^{[m]}$ are $p^{[m]}_{x,x+1} = \omega^{[m]}_x$ and $p^{[m]}_{x+1,x} = 1 - \omega^{[m]}_{x+1}$. For $x \in J^{[m]}$ we denote by $\mathbb{P}^{[m]}_x$ the probability distribution of the chain $X^{[m]}$ when it starts from $X^{[m]}_0 = i$. The above construction is done for all $m \geq 0$.

Let

$$\xi_m := \left(\frac{R^{[m]}}{R^{[m+1]}} \right)^2 ,$$

From (6) the following identity is verified:

$$\forall x \in J^{[m+1]} : \; \omega^{[m]}_x (1 - \omega^{[m]}_{x+1}) = \xi_m \, \omega^{[m+1]}_x (1 - \omega^{[m+1]}_{x+1}) \, . \qquad (7)$$

Now, for all $y \in J^{[m]}$ define

$$\tau^{[m]}_y = \inf\{ n > 0 : X^{[m]}_n = y \} \, ,$$

where as usual $\tau^{[m]}_y = \infty$ when $X^{[m]}_n \neq y$ for all $n > 0$.

Proposition 4 *If $R^{[m]} < R^{[m+1]}$ the chain $X^{[m]}$ is positive recurrent. Moreover it has exponential moment,*

$$\forall \, \theta \in (1, \xi_m^{-1}), \; \forall x, y \in J^{[m]} : \qquad \mathbb{E}_x \left(\theta^{\tau^{[m]}_y} \right) < \infty \, . \qquad (8)$$

Proof: For all $k \geq 2$ it holds the following relation, where we put $x_0 = m + 1 = x_{2k}$:

$$\begin{aligned}
\mathbb{P}^{[m]}_{m+1} \left(\tau^{[m]}_{m+1} = 2k \right) &= \sum_{x_1,\ldots,x_{2k-1} > m+1, \, |x_j - x_{j+1}| = 1} p^{[m]}_{m+1,i_1} p^{[m]}_{x_1,x_2} p^{[m]}_{x_2,x_3} \ldots p^{[m]}_{x_{2k-1},m+1} \\
&= \sum_{m+1 < x_1,\ldots x_{k-1}, |x_j - x_{j+1}| = 1} \prod_{j=0}^{k-1} \omega^{[m]}_{x_j} \left(1 - \omega^{[m]}_{x_j+1} \right) \\
&= \sum_{m+1 < x_1,\ldots x_{k-1}, \, |x_j - x_{j+1}| = 1} \left(\prod_{j=0}^{k-1} \xi_m \omega^{[m+1]}_{x_j} \left(1 - \omega^{[m+1]}_{x_j+1} \right) \right) \\
&= \sum_{x_1,\ldots,x_{2k-1} > 1, |x_j - x_{j+1}| = 1,} \xi^k_m p^{[m+1]}_{m+1,x_1} p^{[m+1]}_{x_1,x_2} p^{[m+1]}_{x_2,x_3} \ldots p^{[m+1]}_{x_{2k-1},m+1} \\
&= \xi^k_m \mathbb{P}^{[m+1]}_{m+1} \left(\tau^{[m+1]}_{m+1} = 2k \right) \, .
\end{aligned}$$

where in the third line we used equality (7). For $k = 1$ we have:

$$\mathbb{P}^{[m]}_{m+1} \left(\tau^{[m]}_{m+1} = 2 \right) = (1 - \omega^{[m]}_{m+1}) + \mathbb{P}^{[m+1]}_{m+1} \left(\tau^{[m+1]}_{m+1} = 2 \right) \, .$$

From the hypothesis $\xi_m = \left(\frac{R^{[m]}}{R^{[m+1]}}\right)^2 < 1$. For $1 < \theta \leq \xi_m^{-1}$ we have:

$$\mathbb{E}_{m+1}^{[m]}\left(\theta^{\tau_{m+1}^{[m]}}\right) = \theta^2 \mathbb{P}_{m+1}^{[m]}\left(\tau_{m+1}^{[m]} = 2\right) + \sum_{k\geq 2} \theta^{2k}\, \mathbb{P}_{m+1}^{[m+1]}\left(\tau_{m+1}^{[m+1]} = 2k\right)$$

$$= \theta^2(1-\omega_{m+1}^{[m]}) + \sum_{k\geq 1} (\theta\xi_m)^{2k}\, \mathbb{P}_{m+1}^{[m+1]}\left(\tau_{m+1}^{[m+1]} = 2k\right)$$

$$\leq \theta^2(1-\omega_{m+1}^{[m]}) + \sum_{k\geq 1} \mathbb{P}_{m+1}^{[m+1]}\left(\tau_{m+1}^{[m+1]} = 2k\right)$$

$$= \theta^2(1-\omega_{m+1}^{[m]}) + \mathbb{P}_{m+1}^{[m+1]}\left(\tau_{m+1}^{[m+1]} < \infty\right) < \infty\,.$$

We have shown that the chain $X^{[m]}$ verifies (8) for $x = y = m+1$. By irreducibility this holds for all $x, y \geq m$.

The $R^{[m]}$-positive recurrence follows directly by this fact. Indeed, since there exists $m^* \geq m$ such that $\xi_m^{-y} \geq y$ for all $y \geq m^*$ we get

$$\begin{aligned}
\mathbb{E}_{m+1}^{[m]}\left(\tau_{m+1}^{[m]}\right) &= \mathbb{E}_{m+1}^{[m]}\left(\tau_{m+1}^{[m]}\mathbf{1}_{\tau_{m+1}^{[m]}\leq m^*}\right) + \mathbb{E}_{m+1}^{[m]}\left(\tau_{m+1}^{[m]}\mathbf{1}_{\tau_{m+1}^{[m]}> m^*}\right)\\
&\leq m^* + \mathbb{E}_{m+1}^{[m]}\left(\xi_m^{-\tau_{m+1}^{[m]}}\mathbf{1}_{\tau_{m+1}^{[m]}> m^*}\right) < \infty\,.
\end{aligned}$$

□

Proposition 5 *If $R^{[m]} < R^{[m+1]}$ then for all $k = 0, ..., m$ we have $R^{[k]} < R^{[k+1]}$ and the chain $X^{[k]}$ is positive recurrent and has exponential moment.*

Proof: From Proposition 4 it suffices to show $R^{[k]} < R^{[k+1]}$ for all $k = 0, ..., m$. Let us do it by contradiction. Assume the property does not hold, then fix j as the bigger k smaller than m where the strict inequality fails. So, we have $R^{[j]} = R^{[j+1]} < R^{[j+2]}$. This implies $h({R^{[j+1]}}^2; j+1, \infty) < 1$, in fact in the contrary we should have $h({R^{[j]}}^2; j, \infty) = h({R^{[j+1]}}^2; j, \infty) = \infty$ which is a contradiction. Now, from $R^{[j+1]} < R^{[j+2]}$ we get $h({R^{[j+2]}}^2; j+1, \infty) > 1$.

By the same argument as the one used in Lemma 3 we will conclude that there exists $s \in ({R^{[j+1]}}^2, {R^{[j+2]}}^2)$ such that $h(s; j+1, \infty) = 1$, that contradicts the maximality of $R^{[j+1]}$. We conclude $R^{[j]} < R^{[j+1]}$ and the property holds. □

Proposition 6 *If $R^{[0]} = R^{[m]}$ for all $m \geq 0$ then the chain $X^{[k]}$ is transient for all $k \geq 1$.*

Proof: In this case we necessarily have $h({R^{[0]}}^2; x, \infty) < 1$, $\forall x \geq 1$ showing the assertion. □

From Propositions 5 and 6 it follows Theorem 1.

Remark 7 *If $R^{[0]} = R^{[m]}$ for all $m \geq 0$ we are not able to classify completely $X^{[0]}$. We can only assert that $h({R^{[0]}}^2; 0, \infty) < 1$ implies that $X^{[0]}$ is transient, and if $h({R^{[0]}}^2; 0, \infty) = 1$ then $X^{[0]}$ is recurrent. But, we cannot state when it is null or positive recurrent.*

Remark 8 *Assume the hypothesis of Proposition 6. Let $0 < s < s^{*[0]}$. Then $0 < h(s; 0, \infty) < 1$. Take $a_{-1} = \frac{1-h(s;0,\infty)}{s}$ then the sequence $\vec{a}^{(-1)} = (a_{-1}, a_0, ..., a_n, ...)$ verifies $h(s; -1, \infty) = \frac{a_{-1}s}{1-h(s;0,\infty)} = 1$, $h(s; x, \infty) < 1$, $\forall x \geq 1$. On the other hand for $s < s' < s^{*[0]}$, $h(s'; -1, \infty) = \frac{a_{-1}s'}{1-h(s';0,\infty)} > 1$. So, $s = s^{*[-1]} < s^{*[0]}$, and the sequence $\vec{a}_{-1}$ has a gap and so the extension of the matrix Q to $\{-1, 0, ...\}$ with $q_{-1,0}q_{0,-1} = a_{-1}$, is $R-$positive.*

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